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# A Report on the Numerical Solution of the Geostrophic Conservation Equation

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# A Report on the Numerical Solution of the Geostrophic Conservation Equation

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## I. Introduction

At present work is progressing under the guidance of E. Isaacson and G. K. Morikawa, and others, aiming at a feasible numerical method of solving the geostrophic conservation equation. It is believed that solutions of this equation may contribute to the understanding of the problem of meteorological forecasting, especially when applied to the problem of predicting the path of large disturbances such as hurricanes.

In this paper we shall deal mainly with describing the numerical methods which went into solving the equations. Other papers presently in preparation will discuss the meteorological implications of results obtained so far. [9]

In the following we shall indicate the physical model used and give a statement of the derivation of the basic equations. For a detailed treatment of this derivation one should consult the paper of reference. [5]

The physical model used is that of a single layer of incompressible, inviscid, homogeneous atmosphere at constant density, with a free surface at a height which includes most of the atmosphere when equivalently reduced to constant density. The height of the free surface is small compared with the average wave length of large scale disturbances, and hence the shallow

water, gravity wave approximations may be used. [11] Alternatively one may use, if necessary, a model of several layers of constant densities with a free surface at the topmost layer and interfaces between layers. In particular we also use the two layer model with a deep layer of less dense warm fluid overlying a relatively thin layer of denser cold fluid.

Since the phenomena under study are local compared with the surface area of the earth, we take instead of a spherical ground surface a plane tangent to the earth, with the point of tangency centered at the locale of interest.

The derivation of the basic equations is from the shallow water gravity wave equations. [6] An exact solution of these equations is given by a free surface of constant height and an atmosphere at rest. Because of the slow movement of large scale atmospheric phenomena this basic solution is appropriate for a perturbation expansion. The quantities used here are the first order terms of a perturbation on this state of rest. The order of the terms in this expansion is selected on a strictly mathematical basis which is implied by the conservation of certain quantities and the assumption of geostrophic flow. The resulting equations for the first order terms of the perturbation expansion are a nonlinear set and are called, collectively, the geostrophic conservation equation. These equations are related to those recently considered by B. Bolin. [12]

## II. Statement of the Mathematical Problem

Set up a rectangular coordinate system in the plane of tangency, with the origin at the point of tangency. Let the x coordinate increase eastward and the y coordinate increase northward. Let the time be t. Set $\tau = \alpha t$, where $\alpha$ is a scaling constant. Let $h^{(1)}(x,y,\tau)$ be the height to first order of the free surface above or below a constant plane surface of height $h_o$. Here $h_o$ is the constant height of the free surface for the flow at rest. The quantity $\psi^{(1)}$ is defined by

$$\psi^{(1)} = \frac{gh^{(1)}}{f} \tag{1}$$

where g is the acceleration due to gravity and f is the Coriolis parameter:

$$f = 2\Omega \sin \Phi \quad . \tag{2}$$

Here $\Omega$ is the angular velocity of the earth and $\Phi$ is the latitude. For convenience, we choose f as sensibly constant, evaluating it at some central point of interest — this is not essential to the method of solution, as it may easily be allowed to vary.

We now wish to study the atmospheric flow for times $\tau$, $0 \leq \tau \leq T$ at a level high enough to exclude the turbulence effects of the ground — say a height corresponding to 500 millibars of pressure (approximately 19 to 20 thousand feet). Let the components of the first-order horizontal velocity at this level be $u^{(1)}(x,y,\tau)$ and $v^{(1)}(x,y,\tau)$. The basic equations of flow are:

$$\frac{d^{(1)}}{d\tau}\left(\nabla^2 \underline{\psi}^{(1)} - k^2 \underline{\psi}^{(1)}\right) = 0$$

(3)
$$u^{(1)} = -\underline{\psi}_y^{(1)}$$

$$v^{(1)} = \underline{\psi}_x^{(1)} \quad .$$

Here letter subscripts denote partial differentiation, and we have:

a) $\nabla^2 \underline{\psi}^{(1)} = \underline{\psi}_{xx}^{(1)} + \underline{\psi}_{yy}^{(1)}$

b) $\frac{d^{(1)}}{d\tau} = (\ )_\tau + u^{(1)}(\ )_x + v^{(1)}(\ )_y$

c) $k^2$ is a constant. $k^2 = \frac{f^2}{gh_o}$ .

The letters f, g, and $h_o$ are as before. We now drop the superscript (1) for convenience and speak only of the first order terms of the perturbation expansion. The system (3) is to be solved under the following conditions:

(4) Domain: The domain, D, of the solution is the rectangle $x_c \le x \le x_d$; $y_c \le y \le y_d$ for times $0 \le \tau \le T$.

Initial Values: $\underline{\psi}(x,y,\tau)$ is given as a function of (x,y) in D for $\tau = 0$.

Boundary Values: $\underline{\psi}(x,y,\tau)$ is given on the boundary for all $\tau$ in $0 \le \tau \le T$. $\nabla^2\underline{\psi}(x,y,\tau)$ is given on the boundary for all $\tau$, only at those points of the boundary for which, at

time $\tau$, the velocity is directed into the domain, D. (Alternately, one could prescribe $\nabla^2\underline{\Psi} - k^2\underline{\Psi}$ in the same manner, since $\underline{\Psi}$ is prescribed on the boundary.)

With these mixed initial and boundary values prescribed the solution of system (3) has been shown to exist and is unique. [10] The restriction specifying the vorticity, $\nabla^2\underline{\Psi}$, on the boundary only at times and points where the fluid enters the domain comes from the following consideration. Since $\underline{\Psi}$ is known at all times on the boundary, the quantity $\nabla^2\underline{\Psi} - k^2\underline{\Psi}$ is known on the boundary whenever $\nabla^2\underline{\Psi}$ is known there. Hence specifying $\nabla^2\underline{\Psi}$ is equivalent to specifying $\nabla^2\underline{\Psi} - k^2\underline{\Psi}$. But equations (3) state that $\nabla^2\underline{\Psi} - k^2\underline{\Psi}$ is at time $\tau$, constant along any particle trajectory. At time $\tau$ if $\nabla^2\underline{\Psi} - k^2\underline{\Psi}$ is specified on the boundary at a point P where the flow is outgoing this specified value would, in general, contradict that computed in the interior of D along the particle trajectory leaving the domain at P. On the other hand, at time $\tau$, $\nabla^2\underline{\Psi} - k^2\underline{\Psi}$ must be specified at a point P on the boundary where the flow is incoming, since it cannot be computed from the interior — in fact, the specified value of $\nabla^2\underline{\Psi} - k^2\underline{\Psi}$ at P determines the value of $\nabla^2\underline{\Psi} - k^2\underline{\Psi}$ at every interior point included in the particle trajectory through P.

For the purpose of studying large scale centered disturbances such as hurricanes we divide the quantity $\underline{\Psi}$ into two parts:

$$\underline{\Psi}(x,y,\tau) = \underline{\Psi}_1(x,y,\tau) + \underline{\Psi}_0(x,y,\tau) \quad . \tag{5}$$

Here $\underline{\Psi}_1$ is an unknown function to be solved for and represents the stream function of the regular or steering flow. $\underline{\Psi}_o$ is derived from the atmospheric vortex theory as the known function:

$$(6) \qquad \underline{\Psi}_o(x,y,\tau) = \frac{-\gamma}{2\pi} K_o(kr_o) \quad .$$

$K_o$ is the second solution of the modified Bessel's equation. It is a positive function with a logarithmic singularity at the origin and dies out exponentially for large argument. $\gamma$ is a positive constant, the strength of the vortex, given in reference 8. And $r_o$ is defined as:

$$(7)\ a. \qquad r_o = \sqrt{(x-x_o)^2 + (y-y_o)^2} \quad .$$

Here $x_o = x_o(\tau)$ and $y_o = y_o(\tau)$ are the coordinates of the center of the vortex, which, from the atmospheric vortex theory, moves according to the ordinary differential equations: [7]

$$(7)\ b. \qquad \frac{dx_o}{d\tau} = -(\underline{\Psi}_1)_y \Big|_{x_o,y_o,\tau}$$

$$\frac{dy_o}{d\tau} = (\underline{\Psi}_1)_x \Big|_{x_o,y_o,\tau} \quad .$$

It is clear from the above that the motion of the vortex is determined by the stream function of the regular or steering flow, $\underline{\Psi}_1$. However, it must be pointed out that the vortex also modifies the steering flow. The function $\underline{\Psi}_o$ is separated out from the overall flow $\underline{\Psi}$ as a known function for the purpose of keeping track of it in calculations which would otherwise wash it out. This will become clear in the sequel when it is realized that in practical computation the spacing between mesh points

represent distances on the order of 200 miles — the center of the vortex must be located much more accurately than this mesh size indicates.

## III. The Numerical Method of Solution

For the purposes of computation we write equation (3) as follows

$$\nabla^2 \underline{\Psi} - k^2 \underline{\Psi} = F(x,y,\tau)$$

(8)

$$\frac{dF}{d\tau} = 0 \quad .$$

Now since $\underline{\Psi} = \underline{\Psi}_o + \underline{\Psi}_1$ and $\underline{\Psi}_o = -\frac{\gamma}{2\pi} K_o(kr_o)$ is a solution of $\nabla^2 \underline{\Psi} - k^2 \underline{\Psi} = \delta(x,y)$, we have $\nabla^2 \underline{\Psi}_o - k^2 \underline{\Psi}_o = 0$ for $x \neq x_o$, $y \neq y_o$. Hence we may write system (3) as:

(9)

a. $\nabla^2 \underline{\Psi}_1 - k^2 \underline{\Psi}_1 = F$

b. $F_\tau - \underline{\Psi}_y F_x + \underline{\Psi}_x F_y = 0$

c. $\underline{\Psi}_o = -\frac{\gamma}{2\pi} K_o(kr_o)$

d. $\frac{dx_o}{d\tau} = -(\underline{\Psi}_1)_y \Big|_{x_o,y_o,\tau}$

$$\frac{dy_o}{d\tau} = (\underline{\Psi}_1)_x \Big|_{x_o,y_o,\tau} \quad .$$

Note that in equation (9b) the velocities $u = -\underline{\Psi}_y$, $v = \underline{\Psi}_x$ are the velocities of the total flow, i.e. $\underline{\Psi}_x = [(\underline{\Psi}_1)_x + (\underline{\Psi}_o)_x]$ and $\underline{\Psi}_y = [(\underline{\Psi}_1)_y + (\underline{\Psi}_o)_x]$. If it were not for these quantities deriving from the <u>total</u> stream function we could solve the equations first for the regular or steering flow and determine the

trajectory of the vortex as an afterthought. But this is not possible, since in (9b) the velocity field due to the vortex enters also into the computation of the steering flow.

We now set up a grid mesh in the rectangular domain D and replace the continuous variables $(x,y)$ by $(x_j,y_i)$ as follows:

$$(10)\quad \text{a.}\quad x_j - x_c = j\triangle x \;; \qquad j = 0,1,2,\ldots,M+1$$
$$x_d - x_c = (M+1)\triangle x$$
$$\text{b.}\quad y_d - y_i = i\triangle y \;; \qquad i = 0,1,2,\ldots,N+1$$
$$y_d - y_c = (N+1)\triangle y$$
$$\text{c.}\quad \tau_k = k\triangle\tau \;; \qquad k = 0,1,2,\ldots,T$$

with $\triangle x$, $\triangle y$, $\triangle\tau$, fixed constants. We now write the functions of continuous variables as functions at the discrete points of the mesh, i.e., we write $f(x_j,y_i,\tau_k)$ as $f^k_{j,i}$. The only quantity we leave as a function of continuous variables is the coordinates of the position of the vortex at discrete time steps k — which we write as $(x_o,y_o)^k$.

The method of continuing information from one time step to the next is the following. The function $(\underline{\Psi}_o)^k_{j,i}$ is determined only from $(x_o,y_o)^k$. Initially we have given $\underline{\Psi}^0_{j,i}$ and the position of the vortex $(x_o,y_o)^0$. Hence we can compute $(\underline{\Psi}_1)^0_{j,i} = \underline{\Psi}^0_{j,i} - (\underline{\Psi}_o)^0_{j,i}$. Then we can compute $F^0_{j,i}$ from $\nabla^2(\underline{\Psi}_1)^0_{j,i} - k^2(\underline{\Psi}_1)^0_{j,i} = F^0_{j,i}$. So that we may assume we have initially given $F^0_{j,i}$, $(\underline{\Psi}_1)^0_{j,i}$ and $(x_o,y_o)^0$. Now suppose we have given at time step k, the quantities $F^k_{j,i}$, $(\underline{\Psi}_1)^k_{j,i}$, $(x_o,y_o)^k$.

1. We first compute $(\underline{\Psi}_o)^k_{j,i}$ from knowledge of $(x_o,y_o)^k$, i.e., $\underline{\Psi}_o = -\frac{\gamma}{2\pi} K_o(\kappa r_o)$. (See section VII).

2. We then compute $\underline{\Psi}^k_{j,i} = (\underline{\Psi}_1)^k_{j,i} + (\underline{\Psi}_o)^k_{j,i}$ and its derivatives $(\underline{\Psi}_y)^k_{j,i}$, $(\underline{\Psi}_x)^k_{j,i}$ using central differences.

3. Now using equation (9b) and the known boundary values of $F^{k+1}_{j,i}$ for incoming flow, we find $F^{k+1}_{j,i}$. (See section IV).

4. Using $F^{k+1}_{j,i}$, the known boundary values of $(\underline{\Psi})^{k+1}_{j,i}$ and equation (9a) we solve for $(\underline{\Psi}_1)^{k+1}_{j,i}$. (See section V).

5. We now find $(x_o,y_o)^{k+1}$ by using $(\underline{\Psi}_1)^k_{j,i}$, $(\underline{\Psi}_1)^{k+1}_{j,i}$ and $(x_o,y_o)^k$. (See section VI).

Hence after step 5, we have $F^{k+1}_{j,i}$, $(\underline{\Psi}_1)^{k+1}_{j,i}$ and $(x_o,y_o)^{k+1}$. We can then continue to the next time step.

## IV. Solution of the Hyperbolic Equation (9b)

(9b) $$F_\tau + uF_x + vF_y = 0 \quad .$$

We replace the time derivative by a forward difference and the space derivatives by a backward difference or a forward difference depending on the sign of the velocity component. [3] That is:

(11) a. $$(F_\tau)^k_{j,i} = \frac{1}{\Delta\tau}\left(F^{k+1}_{j,i} - F^k_{j,i}\right)$$

b. $$u^k_{j,i}(F_x)^k_{j,i} = \frac{1}{\Delta x}\,|u^k_{j,i}|\left(F^k_{j,i} - F^k_{(j-\operatorname{sgn} u^k_{j,i}),i}\right)$$

(11) c. $v^k_{j,i}(F_y)^k_{j,i} = \frac{1}{\Delta y}\,|v^k_{j,i}|(F^k_{j,i} - F^k_{j,(i+\operatorname{sgn} v^k_{j,i})})$

d. $\operatorname{sgn} f = \begin{cases} 1 \text{ for } f \geq 0 \\ -1 \text{ for } f < 0 \end{cases}$ .

Hence solving for $F^{k+1}_{j,i}$ we have:

(12) $$F^{k+1}_{j,i} = \left[1 - \frac{\Delta\tau}{\Delta x}\,|u^k_{j,i}| - \frac{\Delta\tau}{\Delta y}\,|v^k_{j,i}|\right] F^k_{j,i} + \frac{\Delta\tau}{\Delta x}\,|u^k_{j,i}|\,F^k_{(j-\operatorname{sgn} u),i} + \frac{\Delta\tau}{\Delta y}\,|v^k_{j,i}|\,F^k_{j,(i+\operatorname{sgn} v)}.$$

Now assume, only for purposes of estimating the error, that the coefficients $u^k_{j,i}$ and $v^k_{j,i}$ are known functions and hence the equations are linear. We let $(F^k_{j,i} - \bar{F}^k_{j,i}) = \varepsilon^k_{j,i}$, where $\bar{F}^k_{j,i}$ is the true solution of the differential equation. Then

(13) $$\varepsilon^{k+1}_{j,i} = \left[1 - \frac{\Delta\tau}{\Delta x}\,|u| - \frac{\Delta\tau}{\Delta y}\,|v|\right] \varepsilon^k_{j,i} + \frac{\Delta\tau}{\Delta x}\,|u|\,\varepsilon^k_{(j-\operatorname{sgn} u),i} + \frac{\Delta\tau}{\Delta y}\,|v|\,\varepsilon^k_{j,(i+\operatorname{sgn} v)} + O(\Delta\tau^2) + O(\Delta\tau\Delta x) + O(\Delta\tau\Delta y) .$$

Now we note that the error at the (k+1)st step is the weighted average of the error at three points of the k-th step with weights that sum to unity. If each of the weights is non-negative we can conclude:

(14) $$\max_{j,i} |\varepsilon^{k+1}_{j,i}| \leq \max_{j,i} |\varepsilon^k_{j,i}| + O(\Delta\tau^2) + O(\Delta\tau\Delta x) + O(\Delta\tau\Delta y).$$

Now let

$$\varepsilon^k = \max_{j,i} |\varepsilon^k_{j,i}| \quad .$$

Then iterating equation (14) k times we get:

(15) $\varepsilon^k \leq \varepsilon^o + \tau[O(\triangle\tau) + O(\triangle x) + O(\triangle y)]$, since $k\triangle\tau = \tau$.

This gives an estimate for the error at time step k in terms of the initial round-off error and the truncation error. We note that if the initial round-off error vanishes then the error at time $\tau$ tends to zero with $(\triangle x, \triangle y, \triangle\tau)$.

The condition that the weights in equation (13) be positive is satisfied by the computational stability condition:

(16) $$\sqrt{\left(\frac{\triangle\tau}{\triangle x}\right)^2 + \left(\frac{\triangle\tau}{\triangle y}\right)^2} \leq \frac{1}{\max\limits_{i,j,k} \sqrt{(u^k_{j,i})^2 + (v^k_{j,i})^2}} \quad .$$

V. The Solution of the Elliptic Equation (9a)

(9a) $$\nabla^2 \underline{\Psi}_1 - \kappa^2 \underline{\Psi}_1 = F \quad .$$

Assume we are at time $\tau$. For convenience we now reserve the superscript for other purposes and drop the subscript, 1, from $\underline{\Psi}_1$. For the solution of this equation we take the finite difference scheme:

(17) $$\underline{\Psi}_{j+1,i} + \underline{\Psi}_{j-1,i} + \underline{\Psi}_{j,i+1} + \underline{\Psi}_{j,i-1} - (4 + \kappa^2 h^2)\underline{\Psi}_{j,i} = h^2 F_{j,i}$$

where

$$h = \triangle x = \triangle y \quad .$$

When $\underline{\Psi}_{j,i}$ is evaluated on the first or last row or column we take the known boundary values neighboring $\underline{\Psi}_{j,i}$ to the right hand side of equation (17) and denote the right hand side as $G_{j,i}$. Then we may write this system of simultaneous linear equations in the following manner. Let each column of N values of $\underline{\Psi}_{j,i}$ be denoted by

$$\phi_j = \begin{pmatrix} \underline{\Psi}_{j,1} \\ \vdots \\ \underline{\Psi}_{j,N} \end{pmatrix}$$

and the column of M such successive columns by

$$\phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_M \end{pmatrix} \quad .$$

Similarly we set

$$g_j = \begin{pmatrix} G_{j,1} \\ \vdots \\ G_{j,N} \end{pmatrix} \quad ,$$

and

$$g = \begin{pmatrix} g_1 \\ \vdots \\ g_M \end{pmatrix} \quad .$$

Then we note that equation (17) is in the form:

$$a\psi = G \tag{18}$$

where $\psi$ and G are vectors with M·N elements, namely $\{\psi_{j,i}\}$ and $\{G_{j,i}\}$ and a is a huge M·N by M·N matrix. We may write the system more compactly in the form

$$A\phi = g \tag{18a}$$

where A is an M by M matrix with elements that are N by N matrices,

$$A = \begin{pmatrix} B & I & 0 & 0 & \cdots & 0 \\ I & B & I & 0 & \cdots & 0 \\ 0 & I & B & I & \cdots & 0 \\ & & & \cdots & & \\ 0 & 0 & \cdots & I & B & I \\ 0 & 0 & \cdots & 0 & I & B \end{pmatrix}$$

where I is the N by N identity matrix and B is the N by N matrix of numbers.

$$B = \begin{pmatrix} \beta & 1 & 0 & 0 & \cdots & 0 \\ 1 & \beta & 1 & 0 & \cdots & 0 \\ 0 & 1 & \beta & 1 & \cdots & 0 \\ & & & \cdots & & \\ 0 & 0 & \cdots & 1 & \beta & 1 \\ 0 & 0 & \cdots & 0 & 1 & \beta \end{pmatrix} \quad \text{with } \beta = -(4 + k^2h^2) \ .$$

Equation (18) always has a unique solution regardless of the value of k. [4] To show this we consider the homogeneous equations $a\psi = 0$. This is equivalent to equation (17) with $F_{j,i} \equiv 0$, and $\Psi_{j,i} = 0$ on the boundary. Now this equation certainly has the trivial solution. Suppose it had also a

non-trivial solution. Then the value of $|\underline{\Psi}_{j,i}|$ for any interior point is not greater than the average value of $|\underline{\Psi}_{j,i}|$ for its four neighboring points. We can see this from equation (17) and noting $-\beta/4 > 1$. Hence $|\underline{\Psi}_{j,i}|$ is not a positive maximum for any interior point. This contradicts the assumption of the existence of a non-trivial solution. Hence the homogeneous equation $a\psi = 0$ has only the trivial solution. This establishes the unique existence of the solution of (18).

To solve the system (18) we use the following iteration method. [1] First we make a guess for the values of $\underline{\Psi}_{j,i}$. In practice this guess is the value of $\underline{\Psi}_{j,i}$ from the previous time step. Using superscripts now to denote the number of the iterant we call this initial guess $\underline{\Psi}^0_{j,i}$. Now we start at $j = 1$ (the first column) and solve the system:

$$\phi_0^0 + B\phi_1^1 + \phi_2^0 = g_1 \tag{19}$$

for the values

$$\phi_1^1 = \begin{pmatrix} \underline{\Psi}^1_{11} \\ \vdots \\ \underline{\Psi}^1_{1j} \end{pmatrix},$$

using the known boundary values $\phi_0$ and the known (previously guessed) values for $\phi_2^0$. We call the solution $\phi_1^1$ the first iterant for column 1, i.e., $\phi_1 = \phi_1^1$. Then we proceed similarly to solve column 2 using $\phi_1^1 + B\phi_2^1 + \phi_3^0 = g_2$. We solve for $\phi_2^1$, using the freshly computed known values of $\phi_1^1$ and the old values $\phi_3^0$. Proceeding thus, we solve for each column and obtain the first

iterants $\phi_j^1$. Then we proceed in general to solve for higher iterants in a similar manner using:

$$\phi_{j-1}^{n+1} + B\phi_j^{n+1} + \phi_{j+1}^{n} = G_j \tag{20}$$

where $\phi_{j-1}^{n+1}$ and $\phi_{j+1}^{n}$ are known either from previous calculations or are known boundary values. We solve equation (20) in the following manner: If we take the first and third terms of the left side of (20) to the right hand side, (20) is in the form:

$$B\phi_j^{n+1} = S_j^{n+1} \tag{20a}$$

with $S_j^{n+1} = G_j - \phi_{j-1}^{n+1} - \phi_{j+1}^{n}$. Now the matrix B may be factored into two matrices, i.e., B = LU, with L a triangular matrix with zeroes above the principal diagonal; and U also a triangular matrix, but with zeroes below the principal diagonal. Then setting $U\phi_j^{n+1} = \zeta_j^{n+1}$ we have:

$$L\zeta_j^{n+1} = S_j^{n+1} \tag{20b}$$

which is simple to solve for $\zeta_j^{n+1}$, since L is a triangular matrix. Then we have:

$$U\phi_j^{n+1} = \zeta_j^{n+1} \tag{20c}$$

which is again easy to solve for $\phi_j^{n+1}$ since U is a triangular matrix. A factorization of B is given by:

$$U = \begin{pmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 1 & 0 & \cdots & 0 \\ 0 & 0 & a_3 & 1 & \cdots & 0 \\ & & & \cdots & & \\ 0 & 0 & \cdots & 0 & a_{N-1} & 1 \\ 0 & 0 & \cdots & 0 & 0 & a_N \end{pmatrix}$$

(20d)

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{a_1} & 1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{a_2} & 1 & 0 & \cdots & 0 \\ & & & \cdots & & \\ 0 & 0 & \cdots & \frac{1}{a_{N-2}} & 1 & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{a_{N-1}} & 1 \end{pmatrix}$$

where the factors $a_n$ are given recursively by

$$a_{n+1} = \beta - \frac{1}{a_n} \quad , \qquad \text{and} \quad a_1 = \beta \quad .$$

No difficulty in computing these factors can be encountered since the sequence $\{a_n\}$ is a monotonic, increasing sequence with bounds:

(20e) $$-4 < a_n < -2 \qquad \text{for all } n \text{ and } n \quad .$$

To accelerate matters we use the extrapolated Liebmann method and take for the (n+1)st iterant not the solution of equation (20) but a linear combination of the nth iterant and the solution of equation (20), i.e., we solve:

$$\phi_{j-1}^{n+1} + B\bar{\phi}_j^{n+1} + \phi_{j+1}^{n} = G_j \tag{21}$$

for $\bar{\phi}_j^{n+1}$ and then take $\phi_j^{n+1}$ as:

$$\phi_j^{n+1} = (1-a)\phi_j^n + a\bar{\phi}_j^{n+1} \tag{22}$$

where a is an extrapolation constant to be determined below. $1 \leq a < 2$. Now we have a sequence of iterants which if they converge, converge to the solution of equation (18).

If we write equation (18a) in the form:

$$A\phi = C\phi + D\phi = g \tag{23}$$

with

$$C = \begin{pmatrix} B & 0 & 0 & \cdots & 0 \\ I & B & 0 & \cdots & 0 \\ 0 & I & B & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & I & B \end{pmatrix}$$

and

$$D = \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 0 & I \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

and now write $\epsilon^n = (\phi - \phi^n)$, where $\phi^n$ is the nth iterant of equations (21) and (22); we may now write the error committed in the (n+1)st iterant in terms of the error committed in the nth iterant as:

$$\varepsilon^{n+1} = -C^{-1}D\varepsilon^{n} = E\varepsilon^{n} \quad . \tag{24}$$

Here, the matrix E is equal to $-C^{-1}D$ .

The absolute value of the maximum eigenvalue of E is given by: [2]

$$\frac{a^2 \cos^2 (\pi/M + 1)}{[\beta + 2 \cos (\pi/N + 1)]^2} < 1 \tag{25}$$

for all a in $1 \leq a < 2$. This value, and hence the number of iterations required for a certain accuracy, is minimized by taking a as the root of:

$$a^2\cos^2(\pi/M+1) - (a-1)[\beta + 2\cos (\pi/N+1)]^2 = 0 , \quad 1 \leq a < 2. \tag{26}$$

The inequality (25) assures convergence of the iterants to the solution of equation (18).

Experience in actual computation shows that the norm of the error is reduced, on the average, by about 30 per cent at each iteration — requiring, in practical calculations about 20 iterations to produce 6 figure accuracy.

## VI. The Ordinary Differential Equations (7b)

$$\begin{aligned} dx_o/d\tau &= -(\underline{\Psi}_1)_y\Big|_{x_o,y_o,\tau} \\ dy_o/d\tau &= (\underline{\Psi}_1)_x\Big|_{x_o,y_o,\tau} \quad . \end{aligned} \tag{7b}$$

Since we have available from the computations $(\underline{\Psi}_1)^k_{j,i}$ and $(x_o,y_o)^k$, we find $(x_o,y_o)^{k+1}$ in the following manner. We choose the four points of the mesh surrounding the point $(x_o,y_o)^k$.

Using centered differences of $(\underline{\Psi}_1)^k_{j,i}$ we evaluate $(\underline{\Psi}_1)_x$ and $(\underline{\Psi}_1)_y$ for each of these four points. Then we linearly interpolate these four values to find $(\underline{\Psi}_1)_x$ and $(\underline{\Psi}_1)_y$ at the point $(x_o,y_o)^k$. We then compute:

$$\bar{x}_o^{k+1} = x_o^k + \frac{dx_o}{d\tau}\triangle\tau + O(\triangle\tau^2) = x_o^k - \underline{\Psi}^k_{1_y}\triangle\tau + O(\triangle\tau^2)$$

(27)

$$\bar{y}_o^{k+1} = y_o^k + \frac{dy_o}{d\tau}\triangle\tau + O(\triangle\tau^2) = y_o^k + \underline{\Psi}^k_{1_x}\triangle\tau + O(\triangle\tau^2)$$

Then with this approximate position $(\bar{x}_o,\bar{y}_o)^{k+1}$, we find the boundary values $\underline{\bar{\Psi}}_0^{k+1} = -(\gamma/2\pi)K_o(\kappa\bar{r}_o)$ where $\bar{r}_o = \sqrt{(x-\bar{x}_o)^2 + (y-\bar{y}_o)^2}$ and $(x,y)$ are points on the boundary. Hence we may determine the boundary values of $\underline{\Psi}_1^{k+1}$ from the relation $\underline{\Psi}_1^{k+1} = \underline{\Psi}^{k+1} - \underline{\bar{\Psi}}_0^{k+1}$. The boundary values of $\underline{\Psi}_1^{k+1}$ are used to compute $\underline{\Psi}_1^{k+1}$ at the interior points.

We now difference and interpolate, as above, $\underline{\Psi}_1^{k+1}$ at the four points surrounding $(\bar{x}_o,\bar{y}_o)^{k+1}$, thus obtaining $\underline{\Psi}^{k+1}_{1_x}$ and $\underline{\Psi}^{k+1}_{1_y}$ at the point $(\bar{x}_o,\bar{y}_o)^{k+1}$. We then compute finally:

$$x_o^{k+1} = x_o^k - \frac{1}{2}(\underline{\Psi}^k_{1_y} + \underline{\Psi}^{k+1}_{1_y})\triangle\tau + O(\triangle\tau^3)$$

(28)

$$y_o^{k+1} = y_o^k + \frac{1}{2}(\underline{\Psi}^k_{1_x} + \underline{\Psi}^{k+1}_{1_y})\triangle\tau + O(\triangle\tau^3)$$

## VII. Computation of the Function $\underline{\Psi}_o = -(\gamma/2\pi)K_o(\kappa r_o)$

The strength of the vortex, $\gamma$, is taken as constant, although provisions have been made to allow this to vary in time as the computation progresses. It may be calculated from the

initial data by evaluating the numerical analogue of the integral formula: [8]

$$(29) \qquad \gamma = \int_B \underline{\Psi}_n \, ds - k^2 \iint_R \underline{\Psi} \, dxdy$$

where R is any region containing the vortex and B is its boundary. Another procedure is to choose that value of $\gamma$ such that the resulting steering flow computed from $\underline{\Psi}_1 = \underline{\Psi} - \underline{\Psi}_o$ is smooth and has at the position of the vortex its contour lines tangent to the direction of the initial velocity of the vortex.

The computation of the $K_o$ Bessel function as carried out by an automatic computer is effected by evaluating the classical series representation. Since the range of the argument, $kr_o$, is between 0 and 2, only eight terms of the series are needed to get ten digit accuracy. The singularity of the function $K_o$ for zero argument is circumvented by truncating the function when the argument is close to zero. In practice this truncation is taken at a height on the graph of $K_o$ which allows tangential velocities due to the vortex no more than a certain maximum on the order of 100 to 200 miles per hour. This maximum allowable velocity is computed from the computational stability requirement given in equation (14).

## VIII. Computations on the Univac

The procedures outlined here for the numerical computation were effected on the Univac machine at New York University. A coded program of considerable flexibility exists now to perform automatically the computations and test the feasibility of this

method. The code will handle grids of sizes up to 20 by 20 points. It is an especially nice feature of this problem that the variation of the physical quantities involved can be estimated a priori and thus the problem has been scaled accordingly for fixed point arithmetic.

In practice, time increments on the order of one hour with mesh spacing representing distances on the order of 200 miles have been used. The initial data required is obtained by interpolating from the U. S. Weather Bureau's charts of the 500 millibar constant pressure surface. Detailed information concerning physical measurements in the region of a hurricane have been, at the present writing, either non-existent or not obtainable in a readily useful form.

This problem was coded for Univac rather than for a faster IBM machine of greater internal memory storage because one retains greater control of computational flow in such a pilot study. But even with the relatively slow Univac and using the coded program at maximum capacity a 24 hour prediction is obtainable in about 4 to 5 hours. According to our most reliable estimates the IBM 704 can, in a production run, produce a 24 hour prediction, using this method, in about 4 to 5 minutes. This is possible for two reasons. First, the IBM computes ten times faster and secondly a factor of six is saved by handling all data internally in the memory rather than transferring from tape to memory — a time-consuming process required in the Univac program.

## IX. Test Runs

The coded program was tested against a simple exact solution of the geostrophic conservation equation. This served several purposes; first, to test the proper functioning of the code in all its parts, and, secondly, to verify the estimates of the accuracy of computation, and thirdly, to test the proper scaling of physical quantities. The exact solution used is:

$$\underline{\Psi}(x,y,\tau) = -Ae^{ky} + BK_o(\kappa r_o) \qquad A,B \text{ constants} \quad . \tag{30}$$

This is already in the form $\underline{\Psi} = \underline{\Psi}_1 + \underline{\Psi}_o$. We can compute from (7b) the trajectory of the vortex as:

$$\begin{aligned} x_o(\tau) &= x_o(0) + \kappa A\tau = \kappa A\tau \\ y_o(\tau) &= y_o(0) = 0 \quad . \end{aligned} \tag{31}$$

We computed and chose A so that the vortex should move with velocity, $\kappa A$, 20 m.p.h. due east. Using time steps of one hour and mesh width of 200 miles for a 20 by 20 grid we inserted the initial and boundary values appropriate to equation (30). The computational results showed that the vortex did indeed move 20 m.p.h. due east and the steering flow remained unchanged in time, as in equation (30). This calculation was carried out for 24 time steps and the accuracy of the results remained well within the limits designed. Variation in quantities was amply allowed for in the scaling of the physical quantities.

Next, data estimated from a standard Weather Bureau chart was used as initial and boundary data for the code. The results

of this test run for 48 hourly time steps are being prepared and partial results will be analyzed in a later report by G. K. Morikawa.

At present, using the experience gained in the previous tests, we are making a full-scale test of the code on as accurate Weather Bureau data as we can obtain — specifically, choosing the data from an actual hurricane, Ione of 1955. A number of auxiliary routines are being coded to aid in the editing and preparation of both input and output data.

## X. Concluding Remarks

The numerical solution of the geostrophic conservation equation as outlined here is the subject of a forthcoming doctoral dissertation by Mrs. H. Montvila. She discusses, in a rigorous manner, the convergence of the numerical solution to the actual solution of the equations, for a wide class of initial and boundary value data. We have previously mentioned the rigorous existence and uniqueness proofs of C. Sensenig.

The computational results of test runs so far strongly indicate the feasibility in both accuracy and speed of solving the geostrophic conservation equation by numerical methods.

## Notes and References


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